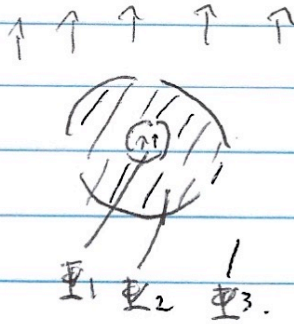
Jackson 4.8 (a)



$$\Phi(r,\theta) = \begin{cases} \Phi_1 & r \leq a \\ \Phi_2 & a \leq r \leq b \\ \Phi_3 & b \leq r \end{cases}$$

Clearly, expanding Φ_1, Φ_2, Φ_3 gives

$$\Phi_1(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l[\cos\theta] \qquad r \leq a.$$

$$\Phi_2(r,\theta) = \sum_{l=0}^{\infty} [B_l r^l + C_l r^{-(l+1)}] P_l[\cos\theta] \qquad a \leq r \leq b$$

$$\Phi_3(r,\theta) = \sum_{l=0}^{\infty} [D_l r^l + F_l r^{-(l+1)}] P_l[\cos\theta] \qquad b \leq r$$

The boundaries at a, b have no free charge, so we match their normal derivatives:

$$\varepsilon_0 \frac{\partial \Phi_1}{\partial r}\Big|_a + \varepsilon\left(-\frac{\partial \Phi_2}{\partial r}\right)\Big|_a = \sigma_b = 0.$$

$$\Rightarrow \quad \varepsilon_0 \frac{\partial \Phi_1}{\partial r}\Big|_a = \varepsilon \frac{\partial \Phi_2}{\partial r}\Big|_a.$$

Similarly, $\displaystyle \varepsilon \frac{\partial \Phi_2}{\partial r}\Big|_b = \varepsilon_0 \frac{\partial \Phi_3}{\partial r}\Big|_b.$

The potential being continuous gives

$$\Phi_1\big|_a = \Phi_2\big|_a, \qquad \Phi_2\big|_b = \Phi_3\big|_b$$

Lastly, the potential at $r \to \infty$ is given by

$$\lim_{r\to\infty} \Phi = -E_0 r\cos\theta$$

By orthogonality of P_l, these constraints on Φ reduces to constraints on each coefficient of P_l:

system of eq.

$$\begin{cases} A_l a^l = B_l a^l + C_l a^{-(l+1)} \\ \varepsilon_0 l A_l a^l = \varepsilon [B_l l a^l - C_l [l+1] a^{-(l+1)}] \\ D_l b^l + F_l b^{-(l+1)} = B_l b^l + C_l b^{-(l+1)} \\ \varepsilon_0 [D_l l b^l - (l+1) F_l b^{-(l+1)}] = \varepsilon [B_l l b^l - C_l [l+1] b^{-(l+1)}] \\ \lim_{r\to\infty} \Phi_3 = -E_0 r\cos\theta \end{cases}$$

We have 5 algebraic variables to solve, and there are 5 ~~system~~ equations of constraint, so the coefficients can be solved.

Examining $\lim_{r \to \infty} \Phi_3 = -E_0 r \cos\theta$ quickly reveals $D_l = 0$ for all l except when $l = 1$, where $D_1 = -E_0$.
This reduces the problem to 4 variables with 4 eq.
It would then be convenient to express all other variables in terms of D_l, since we have solved D_l completely.

The system only involves coefficients with subscript l, so we can drop it. Letting $\varepsilon \equiv \frac{\varepsilon}{\varepsilon_0}$, $\alpha \equiv a^{-(2l+1)}$, $\beta \equiv b^{-(2l+1)}$, we can drastically clean the remaining 4 equations:

$$\begin{cases} A = B + C\alpha \\ lA = \varepsilon\left[Bl + C[l+1]\alpha\right] \\ D + F\beta = B + C\beta \\ lD - (l+1)F\beta = \varepsilon\left[lB - [l+1]C\beta\right] \end{cases}$$

Now we try to write A, B, C, F in D: We will mark important equations in the solving process with $*$.

$$lB + l\alpha C = \varepsilon[lB + [l+1]\alpha C]$$

$*1$ $$(1-\varepsilon)B = \left[\frac{(l+1)}{l}\alpha\varepsilon - \alpha\right]C, \quad (1-\varepsilon)B = \alpha\left[\frac{l+1}{l} - 1\right]C.$$

$$(l+1)D + (l+1)F_\beta = (l+1)B + (l+1)\beta C.$$

$$\Rightarrow (2l+1)D = [\varepsilon l + l + 1]B + (l+1)(1-\varepsilon)\beta C.$$

$\Uparrow$

Substitute $*1$ for this.

$$(2l+1)D = \frac{(\varepsilon l + l + 1)}{1-\varepsilon}\alpha\left[\frac{(l+1)}{l} - 1\right]C + (l+1)(1-\varepsilon)\beta C.$$

$$\Rightarrow (2l+1)D = \left[\frac{\varepsilon l + l + 1}{1-\varepsilon}\alpha\frac{1}{l} + (l+1)(1-\varepsilon)\beta\right]C.$$

$*2$ $$\Rightarrow C = (2l+1)\left[\frac{\varepsilon l + l + 1}{1-\varepsilon}\frac{1}{l}\alpha + (l+1)(1-\varepsilon)\beta\right]^{-1} D.$$

$*2$ is extremely helpful since $D_l = 0$ for $l \neq 1$.
Moreover, $*1$ tells us $B \propto C$, thus $B_l = 0$ for all $l \neq 1$ as well.

By examing the original equation, we have

$$A = B + \alpha C$$

this is in D now because we have B, C in D.

Similarly, we have F in D by the same logic.
Evidently, all these coefficients vanish for $l \neq 1$.

We now compute C_1, B_1, A_1, F_1 explicitly with D_1:

*3 $$C_1 = 3\left[\frac{\varepsilon+2}{1-\varepsilon}\alpha + 2(1-\varepsilon)\beta\right]^{-1} D_1$$

*4 $$B_1 = \alpha\left[\frac{2}{1} - 1\right]\frac{1}{1-\varepsilon} C_1$$

$$= \frac{\alpha}{1-\varepsilon} C_1 = \frac{3\alpha}{1-\varepsilon}\left[\frac{\varepsilon+2}{1-\varepsilon}\alpha + 2(1-\varepsilon)\beta\right]^{-1} D_1$$

*5 $$A_1 = B_1 + \alpha C_1$$

$$= 3\left[\frac{\varepsilon+2}{1-\varepsilon}\alpha + 2(1-\varepsilon)\beta\right]^{-1}\left[\alpha + \frac{\alpha}{1-\varepsilon}\right] D_1$$

*6. $$F_1 = B_1\beta^{-1} + C_1 - D_1\beta^{-1}$$

$$= \left\{3\left[\frac{\varepsilon+2}{1-\varepsilon}\alpha + 2(1-\varepsilon)\beta\right]^{-1}\left[1 + \frac{\alpha}{1-\varepsilon}\beta^{-1}\right] - \beta^{-1}\right\} D_1.$$

Now we substitute back $\frac{\varepsilon}{\varepsilon_0} \equiv \varepsilon$, $\alpha \equiv a^{-(2l+1)}$, $\beta \equiv b^{-(2l+1)}$.

For $l=1$, $\alpha = a^{-3}$, $\beta = b^{-3}$.

$$\left[\frac{\varepsilon+2}{1-\varepsilon}\alpha + 2(1-\varepsilon)\beta\right]^{-1} = \left[\frac{\varepsilon+2\varepsilon_0}{\varepsilon_0-\varepsilon}a^{-3} + 2\left(1-\frac{\varepsilon}{\varepsilon_0}\right)b^{-3}\right]^{-1}$$

For notational simplicity, denote by γ.

$$\frac{1}{1-\varepsilon} = \frac{1}{1-\varepsilon/\varepsilon_0} = \frac{\varepsilon_0}{\varepsilon_0-\varepsilon}.$$

$$\Rightarrow \quad C_1 = 3\gamma D_1 = -3\gamma E_0.$$

$$B_1 = 3a^{-3}\frac{\varepsilon_0}{\varepsilon_0-\varepsilon}\gamma D_1 = -3a^{-3}\frac{\varepsilon_0}{\varepsilon_0-\varepsilon}\gamma E_0.$$

$$A_1 = 3\gamma\alpha\left[1+\frac{1}{1-\varepsilon}\right]D_1 = 3\gamma\alpha\left[\frac{2\varepsilon_0}{\varepsilon_0-\varepsilon}\right]D_1$$

$$= -3a^{-3}\gamma\left[\frac{2\varepsilon_0}{\varepsilon_0-\varepsilon}\right]E_0.$$

$$F_1 = \left\{3\gamma\left[1+\frac{a^{-3}\varepsilon_0}{\varepsilon_0-\varepsilon}b^{3}\right] - b^{-3}\right\}D_1$$

$$= \left\{3\gamma\left[1+\frac{\varepsilon_0}{\varepsilon_0-\varepsilon}\left(\frac{b}{a}\right)^3\right] - b^{-3}\right\}\left[-E_0\right].$$

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